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A comparison of two nonclassical measures, entanglement potential and the negativity of the Wigner function

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Abstract

Two measures of nonclassicality, the entanglement potential and the negativity of the Wigner distribution function defined by the volume of its negative domains, are compared based on an investigation of the nonclassicality for Fock states and Schrödinger cat states in a decoherence process. Both the entanglement potential and the total negative probability are reduced in the linear loss process and the partial negative distribution of the Wigner function is wiped out for large losses while the entanglement potential is always positive. We give a bound condition and find that, though not yet mathematically proven in general, the upper bound of 50% is the maximum allowed loss for the survival of the negative distribution of the Wigner function.

1. Introduction

Nonclassical states play an important role in understanding the fundamentals of quantum physics, and have many applications in quantum information processing [1]. There are various forms of nonclassical behaviour which have been extensively investigated, such as antibunching, sub-Poissonian photon statistics, quadrature phase squeezing, negativity or singularity of the Glauber–Sudarshan P-function, the negativity of the Wigner distribution function (WDF), etc. However, none of the above properties detects nonclassicality infallibly [2]. For example, the squeezed state is usually considered as a typical nonclassical state since its quadrature noise is less than that of the vacuum state, but its Wigner function is regular and positive. Fock states with a large number of quanta, on the other hand, have singular P functions and negative Wigner functions but they exhibit no quadrature squeezing and their antibunching behaviour diminishes when the quantum number increases [3], becoming eventually the same as a coherent state. The Schrödinger cat state $|\alpha\rangle + |-\alpha\rangle$ with $\alpha \gg 1$ is well known as a highly nonclassical state, yet it has Poissonian photon statistics and negligible squeezing. Thus, it is still an open issue to quantitatively describe the nonclassicality for a given state. Several universal approaches to quantify the nonclassicality have been proposed. In the early days, Mandel introduced the so-called Mandel Q-factor to describe the departure of the photon number distribution of the state from Poissonian statistics [4]. In 1987, Hillery *et al* defined 'nonclassical distance' in terms of the trace-norm of the difference between the density operator of the quantum state and that of the nearest classical state to measure the nonclassicality [5]. Later, Lee introduced the nonclassical depth of the radiation [6]. However, these criteria cannot reveal all the various quantum effects of the quantum states and it is difficult to quantify precisely how nonclassical a quantum state is. Recently, a measure named the entanglement potential (EP) for quantifying the nonclassicality of the single-mode optical field has been proposed [7], which is a computable universal measure of nonclassicality.

It is well known that quantum entanglement, as a key resource for quantum information, plays a leading role in quantum optics in studying the fundamentals of quantum mechanics [8]. The relationship between quantum entanglement and nonclassicality has been investigated in many papers. It was pointed out that to obtain an entangled output state with an ideal lossless beamsplitter, a necessary condition is that the input state should be nonclassical [9, 10]. Asboth *et al* proposed a measure of the nonclassicality of the single-mode optical field based on the EP [7], which they defined as the quantum entanglement achieved by a 50/50

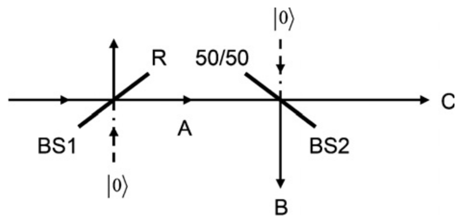

Figure 1. Linear loss system modelled by a beamsplitter BS1. The reflectivity R of BS1 represents the losses that the input state ρ_{in} undergoes; the entangled fields between ports B and C are generated by a 50/50 beamsplitter BS2.

beamsplitter with a nonclassical state from one side and a vacuum state from another side. Another measure named the negativity of the WDF, defined as the volume of the negative parts of the Wigner function, is also exploited as an indicator of nonclassicality [11–14]. It has been shown that, in the case of Fock states $|n\rangle$, the volume of the negative part of the Wigner function increases monotonically with the quantum number n [13]. It has also been suggested to take the absolute value of the total negative probability rather than the minimum negativity of the WDF as an indicator of the quantitative nonclassicality [15].

In this paper, to investigate the nature of the nonclassicality we focus on the question of how robust the nonclassicalities are against the losses. We know that when a nonclassical optical field propagates in a medium, or is detected with imperfect detectors, it inevitably undergoes decoherence and the degree of nonclassicality will be reduced. Let us consider the linear loss process and use a beamsplitter (BS) model to simulate this process [16], as in figure 1. All the losses and imperfect detection can be considered to be the part of the field reflected from the BS. The reflectivity of the BS describes the total losses and the transmitted remaining part gives the total efficiency of the system. Studying the nonclassicality of a generic quantum state in the decoherent environment can essentially test how robust and how large the nonclassicality is. This discussion can help us recognize the profound features of the nonclassical effects and the rationality of this definition of the nonclassicality.

Based on the linear loss process, we have investigated the nonclassicality of Fock states and Schrödinger cat states, the well-known nonclassical states, by exploring their WDF negativity and the EP. It is found that the negative distribution of the Wigner function cannot be present when the losses exceed the bound of 50%, while the EP always exists. We investigate this interesting general bound, and give a bound condition. We also find that the larger the mean photon number of the Fock state or the Schrödinger cat state, the more sensitive the total negative probability is to the linear losses, but this behaviour is not obvious for the EP.

2. WDFs of Fock states and cat states in a linear loss process

Instead of using the density matrix equation for the loss channel, here we consider the linear losses of the BS model, which can describe the photon loss equivalently [17]. Figure 1 describes the process of the nonclassical fields suffering from linear loss, which is represented by input fields passing through the first beamsplitter BS1 with reflectivity R, with R denoting the linear loss. The second beamsplitter BS2 is 50/50 and is set up to generate entangled fields between ports B and C. The dashed arrows stand for the vacuum inputs from the unused ports of the BSs.

We now investigate the behaviour of the WDFs of the nonclassical fields in this process. It has been proved that for an arbitrary quantum state denoted by density operator ρ_{in} impinging on the BS, with a vacuum state coming in from the unused port, the output state can be expressed by [18]

$$\rho_{\text{out}} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left[\frac{1}{m!k!}\left(\frac{R}{T}\right)^{m+k}\right]^{\frac{1}{2}} \times \hat{a}^m T^{\frac{\hat{a}^\dagger\hat{a}}{2}}\rho_{\text{in}}T^{\frac{\hat{a}^\dagger\hat{a}}{2}}(\hat{a}^\dagger)^k \otimes |m\rangle\langle k|. \quad (1)$$

Here we have ignored the dephasing caused by the beamsplitter, but have considered the absorption, mismatching and other linear losses. The state has been represented in the basis of the Fock states, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of the input field and T and R are the transmittance and reflectance of the beamsplitter, respectively. Since the BS1 itself is assumed lossless, we have $R + T = 1$.

When the input state is a Fock state $|n\rangle$ with photon number n, the output state is

$$\rho_{\text{out}} = \sum_{m=0}^{n}\sum_{k=0}^{n}\frac{n!}{\sqrt{m!(n-m)!k!(n-k)!}} \times R^{\frac{m+k}{2}}T^{n-\frac{m+k}{2}}|n-m,m\rangle\langle n-k,k|. \quad (2)$$

Here we have traced over the reflection (loss) and obtained the density matrix of the transmitted field:

$$\rho_n = \sum_{m=0}^{n}\frac{n!}{m!(n-m)!}R^{n-m}T^m|m\rangle_A\langle m|, \quad (3)$$

and the corresponding WDF of the state is given by

$$W_n(q,p,R) = \sum_{m=0}^{n}\frac{n!}{m!(n-m)!}R^{n-m}T^m W_m(q,p), \quad (4)$$

where $W_m(q,p)$ are the Wigner functions of the Fock states [16] with photon number m, which are expressed by

$$W_m(q,p) = \frac{(-1)^m}{\pi}\exp(-q^2-p^2)L_m(2q^2+2p^2). \quad (5)$$

Here $L_m(x)$ denote the Laguerre polynomials. As an example, we show the WDF ($p = 0$) for initial input state $|3\rangle$ for different losses in figure 2, where we see that as the losses increase the negativity of the WDF decreases and fades away until $R =$ 50%. When R increases to 1 the WDF turns out to be that of the vacuum state.

The Schrödinger cat state is a coherent superposition of macroscopically distinguishable quantum states, given by the superposition of two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$, which are separated in phase by 180°. The even Schrödinger cat state is defined as

$$|\text{cat}\rangle = N_e(|\alpha\rangle + |-\alpha\rangle), \quad (6)$$

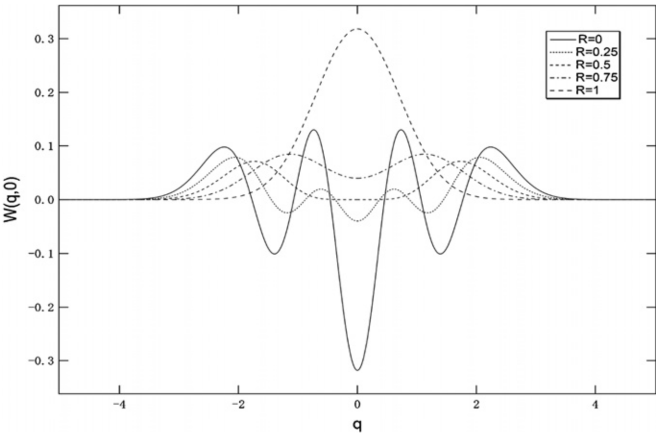

Figure 2. The Wigner distribution function at $p = 0$ for the initial Fock state $|3\rangle$ for various losses.

where $N_e = [2(1 + e^{-2|\alpha|^2})]^{-1/2}$ is the normalization factor. The mean photon number for the cat state (6) is

$$\langle n\rangle_{\mathrm{Cat}} = \frac{1 - e^{-2|\alpha|^2}}{1 + e^{-2|\alpha|^2}}|\alpha|^2. \tag{7}$$

Similarly, we obtain the density matrix of the transmitted field for an initial even Schrödinger cat state (6):

$$\rho_{\mathrm{Cat}} = N_e^2 \exp(-R|\alpha|^2) \sum_{n=0}^{\infty} \times \frac{(R|\alpha|^2)^n}{n!}[|\sqrt{T}\alpha\rangle_A\langle\sqrt{T}\alpha| + (-1)^n|\sqrt{T}\alpha\rangle_A\langle-\sqrt{T}\alpha| + (-1)^n|-\sqrt{T}\alpha\rangle_A\langle\sqrt{T}\alpha| + |-\sqrt{T}\alpha\rangle_A\langle-\sqrt{T}\alpha|]. \tag{8}$$

The corresponding WDF is then obtained to be

$$W_{\mathrm{Cat}}(q, p, R) = 2\frac{N_e^2}{\pi}\exp(-R|\alpha|^2)\exp[-2T|\alpha|^2 - q^2 - p^2] \times \sum_{n=0}^{\infty}\frac{(R|\alpha|^2)^n}{n!}\{\cosh[\sqrt{T}(2\alpha_1 q + 2\alpha_2 p)] + (-1)^n e^{2T|\alpha|^2}\cos[\sqrt{T}(2\alpha_2 q - 2\alpha_1 p)]\}. \tag{9}$$

Here $\alpha_1 = \sqrt{2}\mathrm{Re}(\alpha)$, $\alpha_2 = \sqrt{2}\mathrm{Im}(\alpha)$ and q and p are the real and imaginary parts of the complex amplitude α, respectively. Figure 3 shows the WDFs of the even Schrödinger cat state with $\alpha = 2$ for various losses.

Again, we can see that as the losses increase, the negativity of the WDF fades away, and 50% (~3 dB) loss is the critical threshold above which the WDFs are always positive.

The negativity of the WDF, i.e. the total negative probability of the WDF, can also be used to describe quantitatively the nonclassicality of a given state. The absolute value of the total negative probability P_{NW} is defined as

$$P_{NW} = \left|\int_{\Omega} W(q, p)\,dq\,dp\right|, \tag{10}$$

where Ω is the region of negative WDF. According to the WDFs expressed by equations (4) and (9), we can obtain the total negative probability of the Fock states and the cat states, respectively. Figures 4(a) and (b) show P_{NW} as a function of the losses for the Fock states $|1\rangle$, $|2\rangle$, $|3\rangle$ and cat states with $\alpha = 1, 2, 3$, corresponding to $\langle n\rangle_{\mathrm{Cat}} = 0.76, 4.00, 9.00$, respectively. We find that the larger the mean photon number $\langle n\rangle_{\mathrm{Fock}}$ ($\langle n\rangle_{\mathrm{Cat}}$), the more nonclassical the Fock states (cat states) are, and the more sensitive the nonclassicality is to the linear losses. This

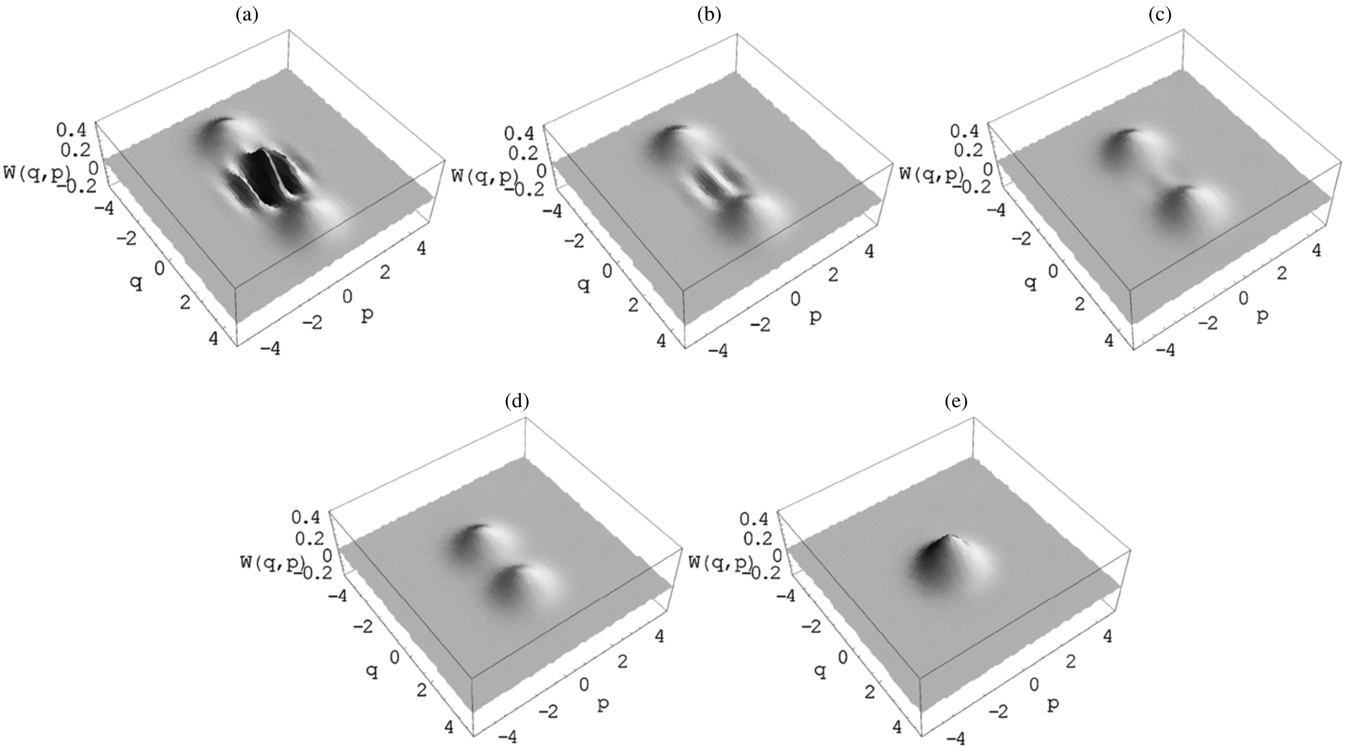

Figure 3. Wigner distribution functions for an even Schrödinger cat state with $\alpha = 2$ in a linear loss process. (a) $R = 0$; (b) $R = 0.25$; (c) $R = 0.5$; (d) $R = 0.75$; (e) $R = 1$.

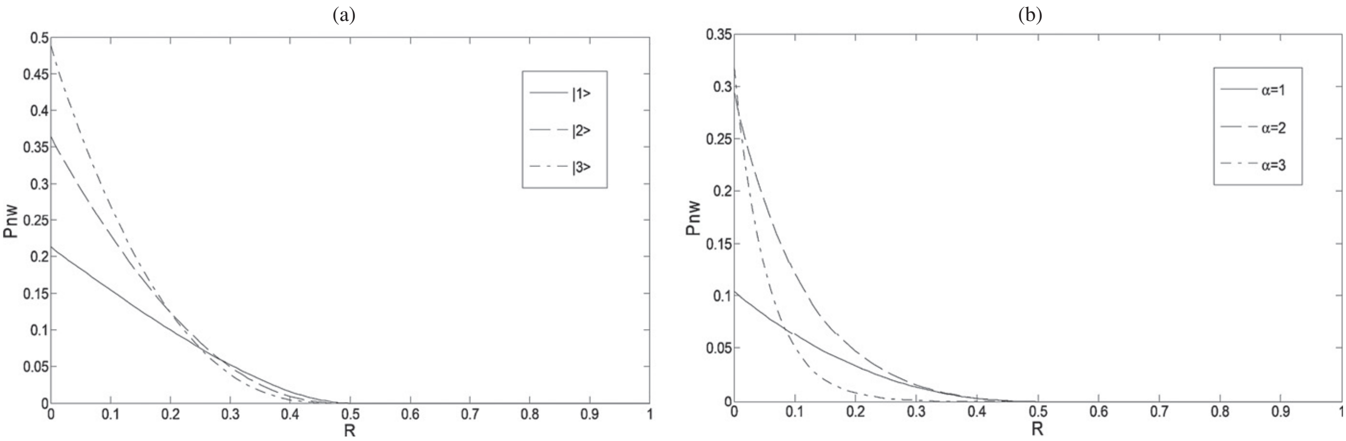

Figure 4. Absolute value of the negative probability of the WDF for (a) Fock states $|1\rangle$, $|2\rangle$, $|3\rangle$ and (b) even Schrödinger cat states with $\langle n\rangle_{\text{Cat}} = 0.76, 4.00, 9.00$, respectively, in a linear loss process.

implies that the more nonclassical the state, the more fragile the state is and the faster the coherence of the state decays [19]. Moreover, we again find that the partial negativity of the WDF disappears for losses larger than $R = 50\%$. This 50% loss seems to be the general bound for the survival of a negative WDF for generic quantum states. This bound of the loss has actually been confirmed not only for the cat states [20], but also for the single-photon-added coherent state and the two-photon-added coherent state [15].

3. EPs of the Fock states and cat states in a linear loss process

Based on the total volume of the negative domains of the WDF, the nonclassicality of Fock states and Schrödinger cat states has been discussed above. This measure of nonclassicality is reasonable but the constraint based on the negative WDF is very strong, as we can see that 50% of the losses would completely kill the negativity. Also, as mentioned above, some quantum states such as the squeezed states are known to be nonclassical but their WDFs are always positive. A relatively weak criterion, which is probably more rational, is the so-called entanglement potential [7]. The main point is to check how much quantum entanglement can be achieved by a given quantum state. We still consider the smeared nonclassical fields which are incident on a 50/50 beamsplitter (BS2, see figure 1) and see how the generated entanglement at the two output ports, B and C, changes along with the losses. The logarithmic negativity [21] is used to describe the degree of the quantum entanglement, which essentially characterizes the EP.

Since we have already obtained the density matrix of the Fock states in the linear loss system shown in equation (3), we now use this mixed state as the input state of the BS2 with vacuum at the other port to generate the entanglement. Here we only consider the initial single-photon state $|1\rangle$ and the two-photon state $|2\rangle$ for the evolution of the EP (the logarithmic negativity). The output field for the initial single-photon state is

$$\rho_{|1\rangle} = R|00\rangle_{BC}\langle 00| + \frac{1-R}{2}|01\rangle_{BC}\langle 01| + \frac{1-R}{2}|01\rangle_{BC}\langle 10| + \frac{1-R}{2}|10\rangle_{BC}\langle 01| + \frac{1-R}{2}|10\rangle_{BC}\langle 10|, \tag{11}$$

and similarly for the two-photon state we have

$$\begin{aligned}\rho_{|2\rangle} = {} & R^2|00\rangle_{BC}\langle 00| + \frac{(1-R)^2}{2}|11\rangle_{BC}\langle 11| \\ & + R(1-R)(|01\rangle_{BC}\langle 01| + |01\rangle_{BC}\langle 10| + |10\rangle_{BC}\langle 01| \\ & + |10\rangle_{BC}\langle 10|) + \frac{(1-R)^2}{4}(|02\rangle_{BC}\langle 02| + |02\rangle_{BC}\langle 20| \\ & + |20\rangle_{BC}\langle 02| + |20\rangle_{BC}\langle 20|) + \frac{(1-R)^2}{2\sqrt{2}}(|02\rangle_{BC}\langle 11| \\ & + |11\rangle_{BC}\langle 02| + |11\rangle_{BC}\langle 20| + |20\rangle_{BC}\langle 11|).\end{aligned} \tag{12}$$

The logarithmic negativity of a bipartite mixed state ρ is defined as [21]

$$LN(\rho) = \log_2 \|\rho^{T_A}\|, \tag{13}$$

where ρ^{T_A} denotes the partial transpose of ρ with respect to subsystem A, and $\|\rho^{T_A}\|$ denotes the trace norm of ρ^{T_A}, which is equal to the sum of the absolute values of the eigenvalues of ρ^{T_A}.

Similarly, for initial Schrödinger cat states (equation (8)), we have

$$\begin{aligned}\rho_{\text{Cat}} = {} & N_e^2\, \mathrm{e}^{-|\alpha|^2} \sum_{h,i,i',j,j'=0}^{\infty} \frac{1}{h!\sqrt{i!i'!j!j'!}} (R|\alpha|^2)^h \\ & \times \left(\sqrt{\frac{1-R}{2}}\alpha\right)^{i+j+i'+j'} [1 + (-1)^{h+i'+j'} \\ & + (-1)^{h+i+j} + (-1)^{i+j+i'+j'}]|i, j\rangle_{BC}\langle i', j'|.\end{aligned} \tag{14}$$

Here we have assumed that α is real, i.e., $\alpha = \alpha^*$ for simplicity. For the continuous variable two-mode entanglement, the logarithmic negativity is a decreasing function of $\tilde{\nu}_-$, and is defined as [21–23]

$$LN(\rho) = \max[0, -\log_2 2\tilde{\nu}_-], \tag{15}$$

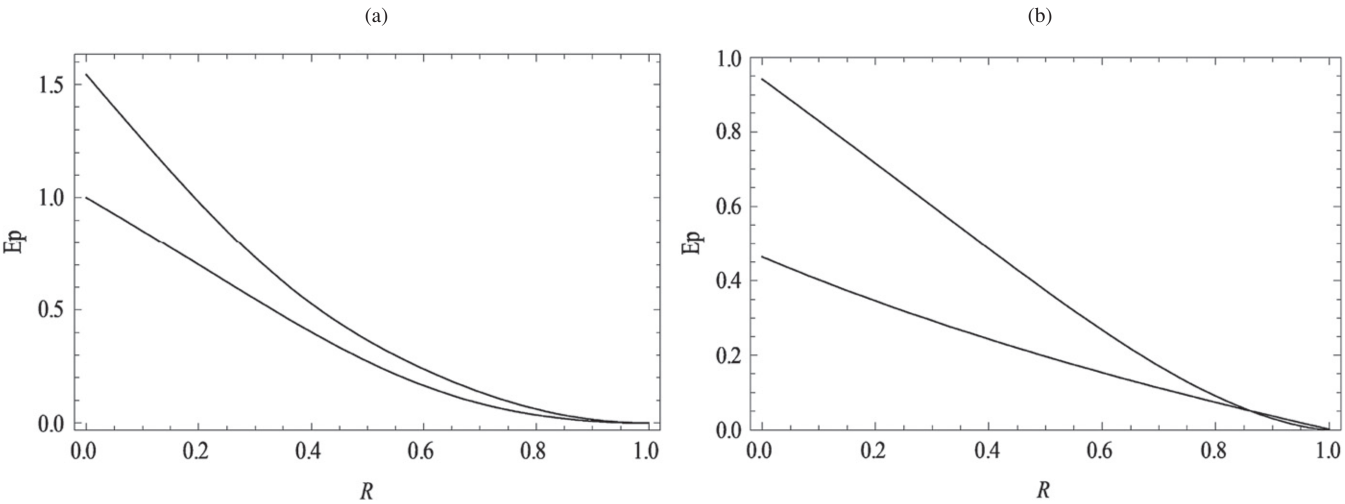

Figure 5. EP as a function of the losses. (a) EP for initial $|1\rangle$ (lower curve) and $|2\rangle$ (upper curve) states as a function of the losses. (b) EP for an initial even Schrödinger cat state with $\langle n\rangle_{\text{Cat}} = 0.76$ (lower curve) and 4.00 (upper curve).

where $\tilde{\nu}_-$ is given by

$$\tilde{\nu}_- = \sqrt{\frac{\tilde{\Delta}(\sigma) - \sqrt{\tilde{\Delta}(\sigma)^2 - 4\mathrm{Det}\sigma}}{2}}, \tag{16}$$

with $\tilde{\Delta}(\sigma) = \mathrm{Det}(A) + \mathrm{Det}(B) - 2\mathrm{Det}(C)$ and the covariance matrix σ being the 2×2 block form

$$\sigma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \tag{17}$$

The elements A, B, C of the covariance matrix σ are given in terms of the conjugate observables x and p in the form [24]

$$A = \begin{pmatrix} \langle x_1^2 \rangle & \left\langle \frac{x_1p_1+p_1x_1}{2} \right\rangle \\ \left\langle \frac{x_1p_1+p_1x_1}{2} \right\rangle & \langle p_1^2 \rangle \end{pmatrix}, \tag{18}$$

$$B = \begin{pmatrix} \langle x_2^2 \rangle & \left\langle \frac{x_2p_2+p_2x_2}{2} \right\rangle \\ \left\langle \frac{x_2p_2+p_2x_2}{2} \right\rangle & \langle p_2^2 \rangle \end{pmatrix}, \tag{19}$$

$$C = \begin{pmatrix} \left\langle \frac{x_1x_2+x_2x_1}{2} \right\rangle & \left\langle \frac{x_1p_2+p_2x_1}{2} \right\rangle \\ \left\langle \frac{x_2p_1+p_1x_2}{2} \right\rangle & \left\langle \frac{p_1p_2+p_2p_1}{2} \right\rangle \end{pmatrix}. \tag{20}$$

Here x_1, x_2, p_1, p_2 are given in terms of the normalized bosonic annihilation (creation) operators a ($a^\dagger$),b ($b^\dagger$) associated with the modes a and b, corresponding to the output fields of port B and C, respectively, and they are defined as

$$\begin{aligned} x_1 &= \frac{a+a^\dagger}{\sqrt{2}}, \qquad p_1 = \frac{a-a^\dagger}{\sqrt{2}i}; \\ x_2 &= \frac{b+b^\dagger}{\sqrt{2}}, \qquad p_2 = \frac{b-b^\dagger}{\sqrt{2}i}. \end{aligned} \tag{21}$$

The final results are shown in figure 5. Figure 5(a) shows the EP of the initial states $|1\rangle$ and $|2\rangle$ as a function of the losses. Clearly, we can see that the logarithmic negativity decreases monotonously with increasing losses, and there is no bound above which the EP is completely wiped out. The larger the photon number, the higher the EP. For small losses, the decrease of the EP is almost linear as the losses increase. Figure 5(b) gives the corresponding results for even Schrödinger cat states with $\alpha = 1$ and $\alpha = 2$, respectively. The EP again decreases linearly as the losses increase even up to about $R = 80\%$, which indicates that the EP could be used to describe the nonclassicality reasonably well. Whatever the states are, the greater the initial EP, the more sensitively is the EP affected by the losses. For different quantum states, the loss-dependent EP is different. Similar to the negative WDF description, in some cases, when the losses exceed a certain value, the amount of nonclassicality of the smeared states with high initial EP may be even less than those states with initially low EP (see figure 5(b)).

It is interesting to compare the above two criteria of the nonclassical descriptions. There are some similarities between the total negative probability P_{NW} of the WDF and the EP (E_p). When the mean photon number increases, both P_{NW} and E_p increase, while they decrease monotonically as the losses increase. However, each of these two descriptions of nonclassicality has its distinctive properties. The total negative probability of the WDF decreases to zero when the losses exceed 50%, while E_p is always positive, approaching zero only for 100% losses. Another feature is that for an initial quantum state with a large mean photon number, P_{NW} decreases more rapidly than E_p with the increase of the losses. This implies that P_{NW} is much more fragile and sensitive to extra losses, and therefore, the criterion of P_{NW} is a stronger constraint for describing the nonclassicality compared with E_p. Although we have only discussed the two criteria based on two typical quantum states in the linear decoherent process, the behaviour is representative. The results show that the WDF and EP provide two constraint standards in the portrayal of nonclassicality, such as the different levels of quantum entanglement [25] or quantum correlations [26]. The EP represents a lower level of nonclassicality of the quantum states compared to the WDF. The stronger the constraint on the nonclassicality, the more fragile and sensitive is the dependence of the corresponding quantum feature on the losses. This is reasonable.

4. Discussion of the WDF for generic quantum states in a linear loss process

It has been shown that the negativity of the WDF disappears as the losses increase to 50%, both for the Fock and the cat states. We now discuss if this bound exists for a generic quantum state with an initially negative WDF.

A generic quantum state ρ can be expanded in the Fock state basis:

$$\rho = \sum_{n,s=0}^{\infty} C_{n,s}|n\rangle\langle s|. \tag{22}$$

The WDF of ρ can be written as

$$W_\rho(q,p) = \sum_{n,s=0}^{\infty} C_{n,s} W_{n,s}(q,p), \tag{23}$$

where $W_{n,s}(q,p)$ denotes the WDF of $|n\rangle\langle s|$. The evolution of $W_\rho(q,p)$ in a linear loss process depends on the evolution of $W_{n,s}(q,p)$ and its coefficient $C_{n,s}$. The WDF of a generic quantum state is defined as [27]

$$W(q,p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \exp(2\mathrm{i}py)\langle q-y|\hat{\rho}|q+y\rangle\,\mathrm{d}y. \tag{24}$$

Here q can be expanded in the Fock state basis ($\hbar = 1$) as [28]

$$|q\rangle = \frac{1}{\pi^{1/4}}\exp\left(-\frac{q^2}{2}\right)\sum_{n=0}^{\infty}\frac{1}{\sqrt{2^n n!}}H_n(q)|n\rangle. \tag{25}$$

Substituting equation (25) into equation (24), we obtain the WDF for $|n\rangle\langle s|$

$$W_{n,s}(q,p) = \frac{1}{\pi^{3/2}}\frac{1}{\sqrt{2^n n!}}\frac{1}{\sqrt{2^s s!}}\int_{-\infty}^{+\infty}\mathrm{e}^{2\mathrm{i}py} \times \exp[-(q^2+y^2)]H_n(q-y)H_s(q+y)\,\mathrm{d}y, \tag{26}$$

where $H_n(x)$ are the Hermite polynomials.

Let us take $|n\rangle\langle s|$ as the input of the beamsplitter BS1, so the output density operator is

$$\rho_{\mathrm{out}} = \sum_{m=0}^{n}\sum_{k=0}^{s}\sqrt{\frac{n!s!}{m!(n-m)!k!(s-k)!}} \times R^{\frac{m+k}{2}}T^{\frac{n+s}{2}-\frac{m+k}{2}}|n-m,m\rangle\langle s-k,k|. \tag{27}$$

Tracing over the loss part, we obtain the density matrix of the transmitted part:

$$\rho_{n,s} = \sum_{m=0}^{\min(n,s)}\sqrt{\frac{n!s!}{(m!)^2(n-m)!(s-m)!}} \times R^m T^{\frac{n+s}{2}-m}|n-m\rangle_A\langle s-m|. \tag{28}$$

Using equation (26), the corresponding WDF of the above density operator $|n\rangle\langle s|$ is derived to be

$$W_{n,s}(q,p,R) = \sum_{m=0}^{\min(n,s)}\sqrt{\frac{n!s!}{(m!)^2(n-m)!(s-m)!}} \times R^m T^{\frac{n+s}{2}-m}W_{n-m,s-m}(q,p). \tag{29}$$

For a Fock state, we obtain

$$W_{n,n}(q,0,R) = \frac{\mathrm{e}^{-q^2}}{\pi}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!} \times R^m(R-1)^{n-m}L_{n-m}(2q^2), \tag{30}$$

where L_{n-m} are the Laguerre polynomials. This shows that around the origin of the phase space, $R = 0.5$ is the boundary between negative and positive WDF. Although this conclusion has not been proved mathematically to be valid for broader quantum states, numerical calculations still help us to find that, for arbitrary $|n\rangle\langle s|$, with $n+s$ being an even number, the negative distributions of the Wigner functions disappear when $R = 0.5$. If either n or s equals 0, i.e. for the forms of $|n\rangle\langle 0|$ or $|0\rangle\langle s|$ ($n, s > 0$), the WDFs have no negative distribution anymore for any amount of loss. For those cases when $n+m$ is odd, there is always a negative distribution of the WDF of $|n\rangle\langle s|$ in the loss process until $R = 1$. As examples, we show the WDFs ($p = 0$) of several initial states $|n\rangle\langle s|$ for 0, 50 and 100% losses in figure 6.

Knowing the evolution of the WDF of $|n\rangle\langle s|$ in the linear loss process, we can thus discuss a generic quantum state with the WDF in general as

$$W_\rho(q,p,R) = \sum_{n,s=0}^{\infty} C_{n,s}W_{n,s}(q,p,R). \tag{31}$$

From the above discussion, we can conclude that for any state

$$\rho = \sum C_{n,s}|n\rangle\langle s|, \tag{32}$$

when $n+s$ is an even number and $C_{n,s}$ is non-negative, there always exists a bound. When the losses exceed this bound of $R = 50\%$, the WDF no longer exhibits negativity. We denote $n+s$ = even and $C_{n,s} \geqslant 0$ as the 'bound conditions' of the state. Similarly, if $n+s$ is odd and $C_{n,s}$ is non-negative, there is no loss bound any more, and the WDF always retains its negativity until $R = 1$.

It is now easy to understand the results discussed in section 2 that for all the Fock states the partial negative distribution of their WDF is wiped out when the losses exceed 50% since all the Fock states satisfy the bound conditions. For an even cat state (6), its density matrix expanded in the Fock state basis is

$$\begin{aligned}\rho_{\mathrm{Cat}} &= N_e^2(|\alpha\rangle\langle\alpha| + |\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|)\\ &= N_e^2\exp(-|\alpha|^2)\sum_{n,s=0}^{\infty}\frac{|n\rangle\langle s|}{\sqrt{n!s!}}[\alpha^n(\alpha^*)^s + \alpha^n(-\alpha^*)^s\\ &\quad + (-\alpha)^n(\alpha^*)^s + (-\alpha)^n(-\alpha^*)^s].\end{aligned} \tag{33}$$

This can be rewritten in the form

$$\rho_{\mathrm{Cat}} = \sum_{n,s=0}^{\infty} C_{n,s}|n\rangle\langle s|, \tag{34}$$

with

$$C_{n,s} = N_e^2\exp(-|\alpha|^2)\frac{\alpha^{n+s}}{\sqrt{n!s!}}[1 + (-1)^s + (-1)^n + (-1)^{n+s}]. \tag{35}$$

Only when both n and s are even will we have $C_{n,s} \geqslant 0$. For all other cases, we have $C_{n,s} = 0$. Here again we have assumed that α is real. What we discussed above belongs to the case that $n+s$ is even and $C_{n,s}$ non-negative, so the negativity of the WDFs of even cat states disappears when the losses exceed 50%.

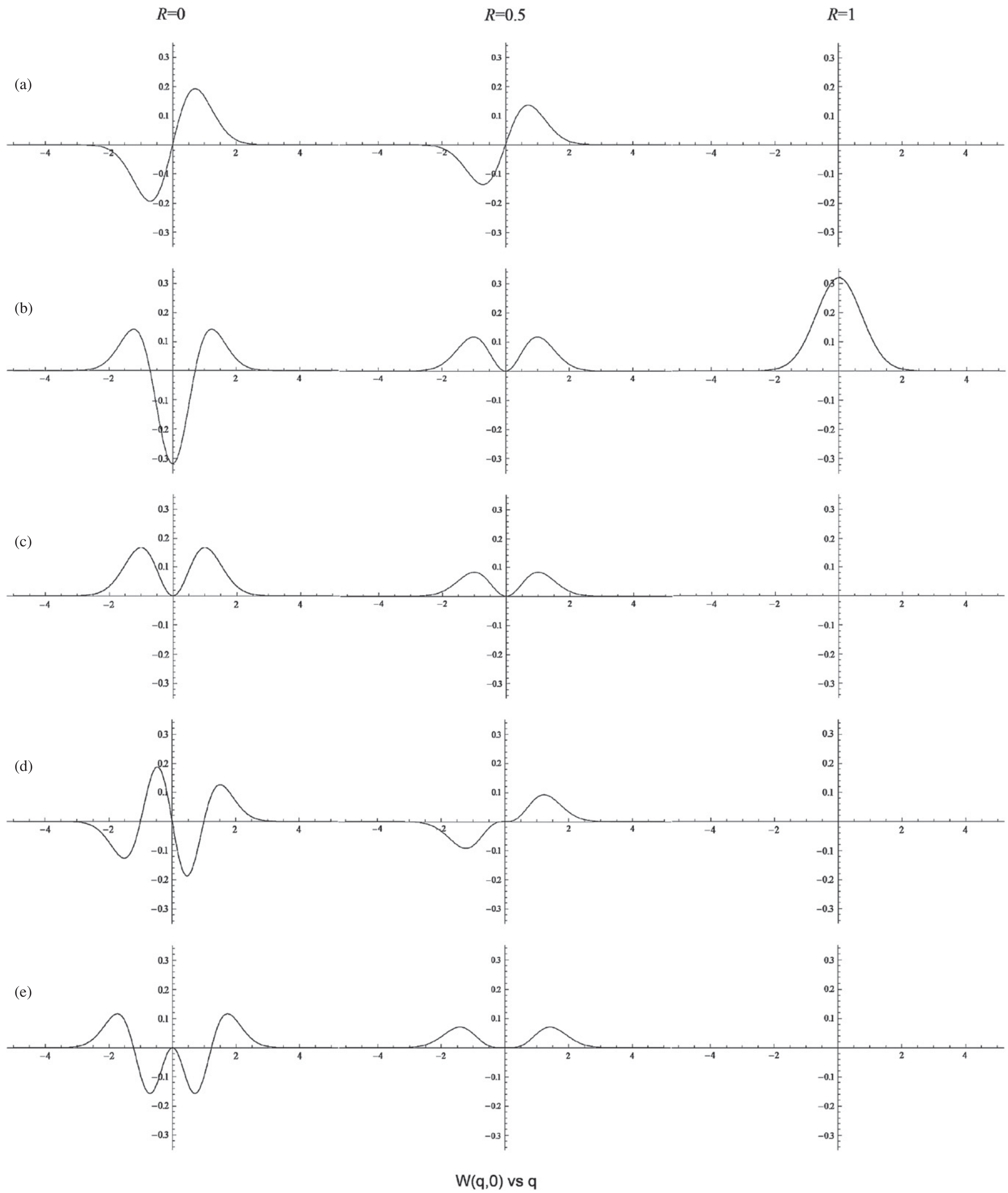

Figure 6. WDF for initial density operators $|n\rangle\langle s|$, (a) $|1\rangle\langle 0|$ (b) $|1\rangle\langle 1|$ (c) $|2\rangle\langle 0|$ (d) $|2\rangle\langle 1|$ (e) $|3\rangle\langle 1|$, for different losses $R = 0$, $R = 0.5$, $R = 1$.

For the odd Schrödinger cat state

$$|\text{cat}\rangle = N_o(|\alpha\rangle - |-\alpha\rangle), \quad (36)$$

with the normalization factor $N_o = [2(1 - e^{-2|\alpha|^2})]^{-1/2}$; its density matrix can also be written in the form of expression (34), with

$$C_{n,s} = N_o^2 \exp(-|\alpha|^2)\frac{\alpha^{n+s}}{\sqrt{n!s!}} \times [1 + (-1)^{s+1} + (-1)^{n+1} + (-1)^{n+s}]. \quad (37)$$

Only when both n and s are odd will we have $C_{n,s} \geqslant 0$. For all other cases, $C_{n,s} = 0$. This also satisfies the bound condition.

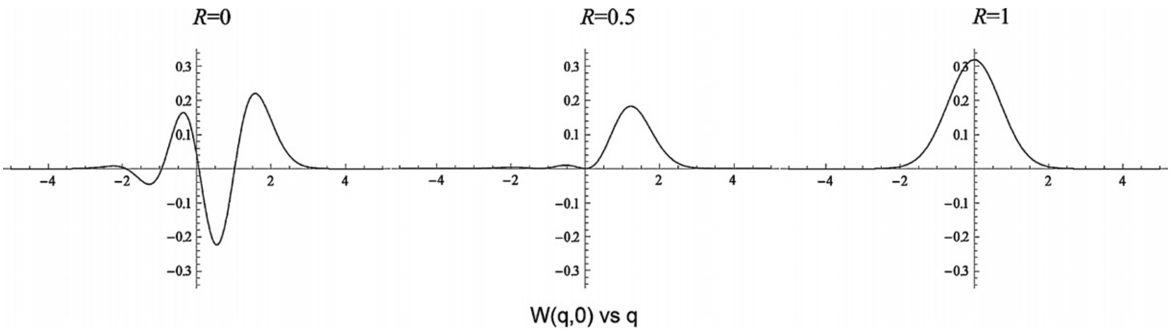

Figure 7. WDF for an initial state given by (38) with $t = 0.75$, for different losses $R = 0$, $R = 0.5$, $R = 1$.

Similar to the even cat states, the WDFs of all the odd cat states with α being real have the same bound in a linear loss process.

It should be noted that the bound conditions are sufficient but not necessary. If a state does not satisfy n+s being even and $C_{n,s}$ non-negative, its WDF can also have a threshold, but all the states satisfying these bound conditions definitely have the bound of losses. As a simple example to show this point, we consider a state composed of a superposition of a single-photon state $|1\rangle$ and a two-photon state $|2\rangle$, i.e.

$$|\varphi\rangle = t|2\rangle + \sqrt{1-t^2}|1\rangle, \tag{38}$$

where t is a real number and $0 \leqslant t \leqslant 1$. Its density operator is

$$\rho_t = t^2|2\rangle\langle 2| + t\sqrt{1-t^2}|2\rangle\langle 1| + t\sqrt{1-t^2}|1\rangle\langle 2| + (1-t^2)|1\rangle\langle 1|. \tag{39}$$

This state does not satisfy the bound conditions, but its WDF also does not have a negative distribution when $R \geqslant 0.5$ as shown in figure 7. In this case, the WDF of the state (39) is a superposition of states $|i\rangle\langle j|$ $(i,j = 1,2)$. Though the WDFs of $|i\rangle\langle j|$ $(i \neq j)$ have a negative distribution when $R = 0.5$ (see figure 6(d)), the WDF of the state composed of all $|i\rangle\langle j|$ $(i,j = 1,2)$ terms can also have no negative distribution. The result depends on the coefficients of $|i\rangle\langle j|$.

5. Summary

We have studied and compared two measures of nonclassicality, the EP and the negativity of the WDF, based on a linear loss system for Fock states and Schrödinger cat states. It is found that both the EP and the total negative probability of the WDF are degraded as the losses increase. However, the partial negative distribution of the WDF is not present for large losses while the EP still exists. The maximum allowed loss for the survival of a negative WDF is 50% for the Fock and cat states. We have discussed this interesting phenomenon, and a general 'bound condition' is given. The WDFs of the states satisfying this condition have no negative distribution when the losses exceed 50%. For larger photon numbers, the nonclassicality is higher and the degradation of the strong constraint criterion of P_{NW}, either for initial Fock states or Schrödinger cat states, is much more sensitive to the extra losses than that of the EP.

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